given number of years. At a future day we hope to be able to give the like particulars as regards all the existing Offices, with such explanatory notes as we may be favoured with. At present the statements issued by many do not supply the necessary information; and of those which do, many are not, while we are writing, accessible. It will be observed that great irregularity exists in the proportions, but that extreme discrepancies are apparent notwithstanding; were the number of quotations increased, we believe the extent of these discrepancies would become better defined, and that the statement would be found useful, as calling timely attention not only to insufficient reserves, but to excessive ones also—the Scylla and Charybdis between which it behoves all engaged in the conduct of these undertakings so carefully to steer.—Ed. *A. M.*

Office.	Years in operation.	Amount assured.	Estimated Liability under Assurances.	Amount Reserved per cent.
		£.	£.	
Alfred	12	601,190	83,223	13·8
Victoria	14	815,171	100,844	12·3
Scottish Provident	15	2,071,125	206,079	9·9
Metropolitan	19	2,753,657	38,671	1·4
Mutual	20	1,017,372	96,918	9·5
Scottish Equitable	22	3,892,031	602,857	15·5
National	23	1,000,000	138,000	13·8
Crown	28	1,940,530	503,185	25·9
Palladium	28	1,280,098	289,933	22·6
Economic	30	5,336,000	1,034,381	19·3
Albion	45	1,443,505	341,176	23·6
Eagle	45	2,723,513	529,616	19·4
London Life	47	6,006,061	715,857	11·9

NOTES AND QUERIES.

Demonstration of Formulæ.—The following demonstration of the expression $\frac{1-\rho A}{1+\rho}$ is perhaps worth recording:—

Since ρ is the annual interest of £1, this last is the present value of a perpetuity of ρ pounds, and $1-\rho A$ is therefore the value of a perpetual reversionary annuity of ρ pounds, to be entered upon at the death of A. But such reversionary annuity is then clearly equal to $1+\rho$. Hence, as $1+\rho : 1-\rho A :: 1 : \frac{1-\rho A}{1+\rho}$, the present value of £1, to be received at the end of the year in which A dies.—Ed. *A. M.*

THE following methods of finding the amount and present value of annuities increasing or decreasing by a constant quantity, are by Mr. EDWIN H. GALSWORTHY, of the Medical and Invalid Assurance Company:—

I. *To find the Amount of an Annuity increasing or decreasing by a Constant Quantity.*—Let the annuity payable at the end of the first year $=a$, and suppose this annuity to increase yearly by any constant quantity, c; then what will the annuity amount to at the end of n years, at i per £ per annum compound interest?

Now the first payment will	be out	at	interest for	$(n-1)$	years,
,, second	,,	,,	,,	$(n-2)$	,,
,, third	,,	,,	,,	$(n-3)$	,,
&c.		&c.		&c.	
nth or last	,,	,,	,,	0	,,

Hence, then,

a	which is payable	at the end	of the 1st year,	will amount to	$a(1+i)^{n-1}$
$a+c$	,,	,,	,, 2nd	,,	$(a+c).(1+i)^{n-2}$
$a+2c$	,,	,,	,, 3rd	,,	$(a+2c).(1+i)^{n-3}$
&c.			&c.		&c.
$a+(n-3)c$	,,	,,	$(n-2)$nd	,,	$\{a+(n-3)c\}.(1+i)^2$
$a+(n-2)c$	,,	,,	,, $(n-1)$th	,,	$\{a+(n-2)c\}.(1+i)$
$a+(n-1)c$	,,	,,	,, nth	,,	$a+(n-1)c$

And the total amount of the increasing annuity S, will be the sum of the last column, or $S=a(1+i)^{n-1}+a(1+i)^{n-2}+\ldots\ldots+a$. (α)
$+c(1+i)^{n-2}+2c(1+i)^{n-3}+\ldots\ldots+c(n-1)$.

Now line (α) represents the amount of an ordinary annuity of £a say, $=$

$A=a.\dfrac{x^n-1}{x-1}$, where x is put for $(1+i)$ . . . . . (β)

Then we have

$$S=A+cx^{n-2}\left\{1+\frac{2}{x}+\frac{3}{x^2}+\frac{4}{x^3}+\ldots\ldots\frac{n-2}{x^{n-3}}+\frac{n-1}{x^{n-2}}\right\}.$$

Let $\dfrac{1}{x}=z$; then the series written within the brackets is $1+2z+3z^2+\ldots$ $\ldots(n-1)z^{n-2}$, which is a series to $(n-1)$ terms, the sum being*

$$=\frac{1-nz^{n-1}+(n-1)z^n}{(1-z)^2}.$$

But we put z for $\dfrac{1}{x}$; therefore,

* This series, or a similar one, is summed in almost every Algebra (*see* Wood's, 12 Edit., Art. ccxcv., &c.), and in the following manner:—

Let $1+2z+3z^2+\ldots\ldots(n-1)z^{n-2}=S$

$\therefore z+2z^2+\ldots\ldots(n-2)z^{n-2}+(n-1)z^{n-1}=z.S$

$\therefore S-z.S=1+z+z^2+\ldots\ldots z^{n-2}-(n-1)z^{n-}$

or, $S(1-z)=\dfrac{1-z^{n-1}}{1-z}-(n-1)z^{n-1}$

$\therefore S=\dfrac{1-z^{n-1}}{(1-z)^2}-\dfrac{(n-1)z^{n-1}}{(1-z)}=\dfrac{1-nz^{n-1}+(n-1)z^n}{(1-z)^2}.$

$$S=A+cx^{n-2}\left\{\frac{1-\frac{n}{x^{n-1}}+\frac{n-1}{x^n}}{\left(1-\frac{1}{x}\right)^2}\right\}=A+c.\left\{\frac{x^{n-2}-\frac{n}{x}+\frac{n-1}{x^2}}{\left(1-\frac{1}{x}\right)^2}\right\}$$

$$=A+c\left\{\frac{x^n-nx+n-1}{(x-1)^2}\right\}=A+\frac{c}{x-1}\left\{\frac{x^n-1}{x-1}\right\}-\frac{nc}{(x-1)^2}.(x-1)$$

and by $(\beta)=A+\frac{c}{x-1}.\frac{A}{a}-\frac{nc}{x-1}=A\left\{1+\frac{c}{(x-1)a}\right\}-\frac{nc}{x-1}$.

But x was put for $(1+i)$;

$$\therefore S=A+\frac{Ac}{ia}-\frac{nc}{i}\text{, and finally}=A+\frac{c}{i}\left\{\frac{A}{a}-n\right\}.$$

When the constant $c=1$ and a also $=1$, we have $S=A+\frac{A-n}{i}$. If the annuity be a decreasing one, write $-c$ for c, and we get the amount $S=A-\frac{c}{i}\left\{\frac{A}{a}-n\right\}$; and when, as before, $c=1$ and $a=1$, then $S=A-\frac{A-n}{i}$.

Example.—Find the amount of an annuity of £1 increasing £1 annually, in 5 years, at 5 per cent.

Here $c=1$ and $a=1$, hence S, or the amount$=A+\frac{A-5}{\cdot 05}$.

(Jones, vol. i.) A= 5·525631
5·
·05)·525631
10·5126= $\frac{A-5}{\cdot 05}$
5·5256= A
16·0382= Amount required.

Proof.—In order to prove the formula, the above result should evidently equal that produced by adding together the amounts of the several payments when accumulated, for the terms during which each would respectively be invested; thus:—

£1	would be invested	4	years,	and would amount to		1·2155
£2	„ „	3	„	„	1·1576 × 2=	2·3152
£3	„ „	2	„	„	1·1025 × 3=	3·3075
£4	„ „	1	„	„	1·0500 × 4=	4·2000
£5	„ „	0	„	„		5·0000
					Sum =	16·0382 as before.

II. *To find the Present Value of an Annuity increasing or decreasing by a Constant Quantity.*—Let the annuity payable at the end of the first year$=a$, and suppose the yearly increase to equal any constant quantity c;

then what is the present value of the annuity payable for n years at i per £ per annum compound interest?

Now at the end of the 1st year a is payable,

,,	,,	2nd ,,	$a+c$,,
,,	,,	3rd ,,	$a+2c$,,
&c.	&c.		&c.
,,	,,	nth ,,	$a+(n-1)c$;

and $1+i$ being the amount of £1 in one year, we obtain the total present value of the increasing annuity, which call P, thus:—

$$P=\frac{a}{(1+i)}+\frac{a+c}{(1+i)^2}+\frac{a+2c}{(1+i)^3}+\ldots\ldots\frac{a+(n-1)c}{(1+i)^n}$$

$$\therefore P=\left\{\frac{a}{(1+i)}+\frac{a}{(1+i)^2}+\frac{a}{(1+i)^3}+\ldots\ldots\frac{a}{(1+i)} \quad . \quad . \quad (\alpha)\right.$$

$$\left.+\frac{c}{(1+i)^2}+\frac{2c}{(1+i)^3}+\ldots\ldots\frac{(n-1)c}{(1+i)^n}\right\} \quad . \quad . \quad (\beta)$$

The series (α) represents the present value of an ordinary annuity of $£a=a.\dfrac{1-(1+i)^{-n}}{i}=a.p$, say; then in the series (β) let $\dfrac{1}{1+i}=x$, and it becomes $=cx^2\{1+2x+3x^2+\ldots.(n-1)x^{n-2}\}$; and summing the series within the brackets (as in the preceding investigation of the amounts), we get

$$\text{Series }(\beta)=cx^2.\frac{1-nx^{n-1}+(n-1)x^n}{(1-x)^2}=c.\frac{x^2-nx^{n+1}+(n-1)x^{n+2}}{(1-x)^2} \quad ..(\gamma)$$

Now p (from above)$=\dfrac{1-(1+i)^{-n}}{i}=\dfrac{1-x^n}{i}$; $\therefore x^n=1-ip$, $\therefore x^{n+1}=x-ipx$, and $x^{n+2}=x^2-ipx^2$; and hence, by substituting these values, we have

$$(\gamma)=c.\frac{x^2-nx+nipx+(n-1)x^2-(n-1)ipx^2}{(1-x)^2}$$

$$=c.\frac{(n+ip-nip)x^2-(n-nip)x}{(1-x)^2}.$$

Here restore the value of x, and the above value becomes

$$c.\frac{(n+ip-nip)(1+i)^{-2}-(n-nip)(1+i)^{-1}}{1-2(1+i)^{-1}+(1+i)^{-2}}.$$

Multiplying numerator and denominator by $(1+i)^2$,

$$=c.\frac{(n+ip-nip)-(n-nip)(1+i)}{(1+i)^2-2(1+i)+1}=c.\frac{ip-ni+ni^2p}{i^2}=c.\frac{p(ni+1)-n}{i};$$

and hence $(\alpha)+(\beta)$, or the total value of $P=ap+c.\dfrac{p(ni+1)-n}{i}$; where $a=1$ and $c=1$, $P=p+\dfrac{p(ni+1)-n}{i}$.

If the annuity be a decreasing one, then the expression for the present value will evidently be represented by the above, after changing the connecting sign $+$ to the sign $-$.

Example.—Find the present value of an annuity of £1 for 5 years, increasing £1 annually, at 5 per cent.

Here $a=1$, $c=1$, $n=5$, $i=\cdot05$, $P=p+\frac{p(ni+1)-n}{i}$.

$$\begin{array}{rl}
\text{(Jones, vol. i.) } p= & 4\cdot329477 \\
 & 1\cdot082369 \\
\hline
 & 5\cdot411846=p(ni+1)=p(1\cdot25) \\
n= & 5\cdot \\
\hline
i=\cdot05) & \cdot411846 \\
\hline
 & 8\cdot23692 \\
p= & 4\cdot32947 \\
\hline
 & 12\cdot56639=\text{Total present value of the increasing annuity.}
\end{array}$$

To check this result, we find the present value of the whole annuity by summing the present values of each distinct payment; and we have—

Years hence.		Present Value of £1.		Amount of Payment.		Present Value.
1	=	·952381	×	1	=	·95238
2	=	·907029	×	2	=	1·81406
3	=	·863838	×	3	=	2·59151
4	=	·822702	×	4	=	3·29081
5	=	·783526	×	5	=	3·91763
Total present value, as before					=	12·56639

which proves the correctness of the formula obtained.

FOREIGN INTELLIGENCE.

DENMARK.—*Statistics of the Royal Octroied Fire Insurance Company, at Copenhagen (Kyl Octroierede Assurance Company for Varer og Effecter).*—This Company, established on the 11th May, 1778, and remodelled on the 24th May, 1843, has a privilege for insuring goods, furniture, and all moveable property, at Copenhagen, so that neither the formation of another Danish Society nor the agency of a foreign Insurance Company is permitted; nevertheless, some English and German Societies have agents at Copenhagen, and do a large amount of business. A fine of 1,000 rix dollars* (£111 sterling) which must be paid by the insured to the Royal Fire Insurance Company, in case he is found to have insured with another Office, has only the effect that this fine is very often insured with the other property; and no inquiry being allowed about the insurance, except in case of fire, the fine can only then be levied.

The original capital amounted to £66,666 sterling, in 500 shares of £56, wholly paid up; but in 1843 it was provided that the reserved fund accumulated at that period, of £4,444, should be added to the original

* 9 Rix dollars = £1.